The Gruenberg-Kegel graph of finite solvable cut groups

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Universidad de Murcia

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Non-Commutative Rings and Applications VII

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Trivial units: $\pm G$.

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4. *For every $\chi \in \mathrm{Irr}(G)$, the field $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) : g \in G)$ is contained in an imaginary quadratic field [Ferraz].*
5. *For every $g \in G$, the field $\mathbb{Q}(\chi(g) : \chi \in \mathrm{Irr}(G))$ is contained in an imaginary quadratic field. [Bächle-Caicedo-Jespers-Maheshwary, 2021].*

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- G is rational if and only if for every $g \in G$ every generator of $\langle g \rangle$ is conjugate to g.
- Every rational group is cut.
- Every symmetric group is rational.
- While only 0.57% of all groups up to order 512 are rational, 86.62% are cut

The Gruenberg-Kegel graph

Gruenberg-Kegel graph = GK-graph = Prime graph:
G non-necessarily finite group.

$$\Gamma_{\mathrm{GK}}(G) : \begin{cases} \text{Vertices: } \pi(G) = \{|g| : g \in G, |g| \text{ prime}\}; \\ \text{Edges: } p - q \text{ with } p \neq q, pq = |g| \text{ for some } g \in G. \end{cases}$$

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1. Every graph is the GK-graph of some group.
2. If G is a finite group then $\Gamma_{\mathrm{GK}}(G)$ has at most 6 connected components [Williams 81, Kondrat'ev 90].
3. Classification of GK-graphs of finite solvable groups [Gruber-Keller-Lewis, 2015].

The Prime Graph Question

$$V(\mathbb{Z}G) = \{\text{Units of } \mathbb{Z}G \text{ with augmentation } 1\}.$$

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Theorem (Kimmerle-Konovalov, 2015)

(PQ) holds for G if and only if it holds for every almost simple epimorphic image of G.

(PQ) has been proved for many almost simple groups including symmetric and alternating groups and several sporadic simple groups [Bächle, Margolis, Konovalov, Bovdi, ...].

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- Classify the GK-graph of solvable cut groups and solvable rational groups.
- Study (PQ) for cut groups and rational groups.

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Known facts

- If G is a rational and solvable then $\pi(G) \subseteq \{2, 3, 5\}$ [Gow, 1976].

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- Classify the GK-graph of solvable cut groups and solvable rational groups.
- Study (PQ) for cut groups and rational groups.

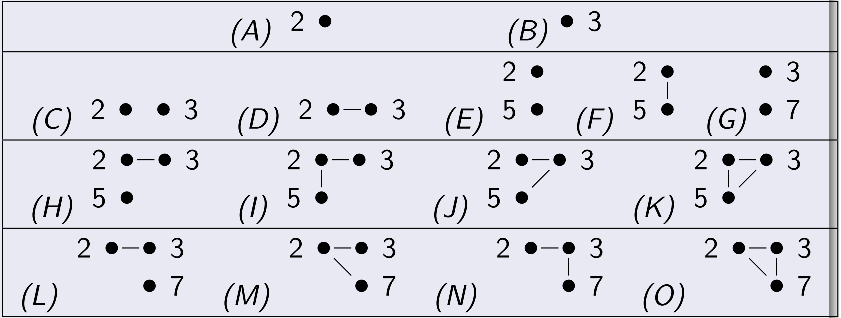
Known facts

- If G is a rational and solvable then $\pi(G) \subseteq \{2, 3, 5\}$ [Gow, 1976].
- If G is a cut and solvable then $\pi(G) \subseteq \{2, 3, 5, 7\}$ [Bachle, 2018].

GK-graphs of finite solvable cut groups: At most 3 vertices

Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

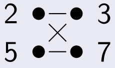
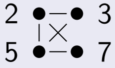
GK-graphs of non-trivial solvable cut groups with at most 3 vertices:



GK-graphs of finite solvable cut groups: More than 3 vertices

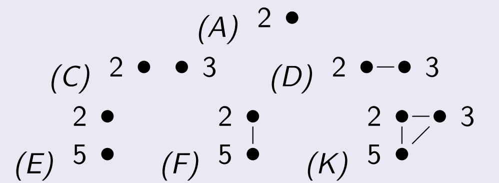
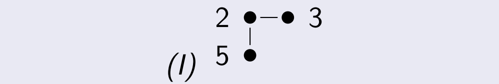
Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

Possible GK-graphs of finite solvable cut groups with more than 3 vertices.

Verified	*(P)*  *(Q)*  *(R)* 
Possible	*(S)*  *(T)*  *(U)*  *(V)* 

Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

<u>Possible</u> GK-graphs of non-trivial solvable rational groups:

Verified	
Possible	

Application 1: GK-graphs of supersolvable rational groups

Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

The following are equivalent for a graph Γ.

1. $\Gamma = \Gamma_{\mathrm{GK}}(G)$ *for some non-trivial metacyclic rational group G.*
2. $\Gamma = \Gamma_{\mathrm{GK}}(G)$ *for some non-trivial metabelian rational group G.*
3. $\Gamma = \Gamma_{\mathrm{GK}}(G)$ *for some non-trivial supersolvable rational group G.*
4. $\Gamma = \Gamma_{\mathrm{GK}}(G)$ *for some non-trivial nilpotent-by-abelian rational group G.*
5. Γ *is one of the graphs (A), (C) or (D).*

Application 2: GK-graphs of supersolvable cut groups

Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

The following are equivalent for a graph Γ.

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2. $\Gamma = \Gamma_{\mathrm{GK}}(G)$ *for some non-trivial metabelian cut group* G.
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4. $\Gamma = \Gamma_{\mathrm{GK}}(G)$ *for some non-trivial nilpotent-by-abelian cut group* G.
5. Γ *is one of the graphs (A) – (G) or (J) – (O).*

In a Monster-Free World

Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

(PQ) holds for cut groups without an epimorphism image isomorphic to the monster group.

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Corollary

(PQ) holds for rational groups.

Thanks for your attention!
Merci pour votre attention!
Ilginiz için teşekkürler
¡Gracias por su atención!